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Pseudo-valuation Rings, II.

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Sunto. – *Viene data una condizione sufficiente affinchè un sopra-anello di un anello di pseudo-valutazione* (*PVR*) *sia ancora un PVR. Da ciò segue che se* (R, M) *è un PVR, allora ogni sopra-anello di R è un PVR se (e soltanto se)* $R[u]$ *è quasi-locale per ciascun elemento u di* $(M: M)$. *Vari risultati sono dimostrati per un ideale primo di un anello commutativo arbitrario R, avente* $Z(R)$ *come insieme di zero-divisori. Per esempio, se P è un primo «forte» di R e contiene un elemento non-zero divisore di R, allora* $(P: P)$ *è un sopra-anello di R con l'insieme degli ideali totalmente ordinato e con ideale massimale P; inoltre,* $(P: P)$ *è un PVR il cui ideale massimale è un ideale primo anche in R se e soltanto se P e* $Z(R)$ *sono entrambi ideali primi «forti» di R. Se* (R, M) *è un PVR, viene dimostrato anche che* $Z(R)$ *può coincidere con* $\mathrm{nil}(R)$ *oppure con un ideale primo propriamente contenuto tra questi due ideali.*

1. – Introduction.

We assume throughout that all rings are commutative with $1 \neq 0$. This paper continues our study of pseudo-valuation rings (as introduced in [6]). We begin by recalling some background material. As in [10], an integral domain R, with quotient field K, is called a *pseudo-valuation domain* (PVD) in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. In [6], we generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [6] that a prime ideal P of a ring R is said to be *strongly prime* (in R) if aP and bR are comparable for all a, $b \in R$. A ring R is called a *pseudo-valuation ring* (PVR) if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [6, Lemma 1(b)]; a chained ring is a PVR [6, Corollary 4]; an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [5, Proposition 3]); and if R is a PVR whose maximal ideal M contains a non-zerodivisor, then $V := (M: M)$ is a chained ring with maximal ideal M [6, Theorem 8].

The following notation will be used throughout. Let R be a ring. Then $Z(R)$ denotes the set of zerodivisors of R, and $\mathrm{nil}(R)$ denotes the set of nilpotent elements of R. Also, $S := R - Z(R) = \{x \in R \mid x \text{ is a non-zerodivisor of } R\}$, $T = R_S$ is the total quotient ring of R, R' denotes the integral closure of R in T, and $U(R)$ denotes the set of units of R. As usual, we say that a ring B is an *overring* *of* R if $R \subset B \subset T$; if I is an ideal of R, then $(I: I) = \{x \in T \mid xI \subset I\}$ is an overring of R and $I^{-1} = \{x \in T \mid xI \subset R\}$; and (R, M) denotes that R is quasilocal with maximal ideal M. Any unexplained material is as in [6], [12].

This paper is organized as follows. Section 2 develops a characterization of the PVRs all of whose overrings are PVRs. Section 3 is devoted to a number of results and examples concerning strongly prime ideals, with, as expected, interplay with the PVR concept. Two typical results in this regard are the following part of Theorem 3.6: if a strongly prime ideal P contains a non-zerodivisor, then $(P: P)$ is a chained ring with maximal ideal P; and Corollary 3.13: if $P \in \mathrm{Spec}(R)$, then $(P: P)$ is a PVR whose maximal ideal is in $\mathrm{Spec}(R)$ if and only if P and $Z(R)$ are both strongly prime ideals of R. Moreover, Example 3.16(c) shows that if (R, M) is a PVR, then $Z(R)$ can be $\mathrm{nil}(R)$, M, or a prime ideal properly contained between these two ideals.

2. – PVRs **whose overrings are** PVRs.

Our first result is a partial converse to the fact that PVRs are quasilocal.

THEOREM 2.1. – Let (R, M) be a PVR and $u \in V - R$. Then $R[u]$ is a PVR if and only if $R[u]$ is quasilocal.

PROOF. – The «only if» assertion is immediate since a PVR is quasilocal [6, Lemma 1(b)].

Conversely, suppose that $R[u]$ is quasilocal. It suffices by [6, Theorem 7] to show that M is the unique maximal ideal of $R[u]$. If $u \notin U(R[u])$, then $u + 1 \in U(R[u]) = U(R[u+1])$ since $R[u]$ is quasilocal, whence $(u+1)^{-1} \in R'$ by [12, Theorem 15]. Hence $(u+1)^{-1} \notin M$ since M is a proper ideal of $R[u]$. Since R' is a PVR with maximal ideal M by [6, Theorem 19], and $(u+1)^{-1} \in R' - M = U(R')$, we have $u + 1 \in R'$ and $u = (u+1) - 1 \in R'$. On the other hand, if $u \in U(R[u])$, then [12, Theorem 15] gives $u^{-1} \in R'$; as $u^{-1} \notin M$ (since M is a proper ideal of $R[u]$), we have $u^{-1} \in R' - M = U(R')$. Thus, in both cases, $u \in R'$, and so $R[u] \subset R'$.

Consider $v \in R[u] - M$. As $v \in R' - M = U(R')$, $v^{-1} \in R'$ and so, by [12, Theorem 15], $v^{-1} \in R[v] \subset R[u]$. In particular, $v \in U(R[u])$. Hence M is the maximal ideal of $R[u]$. ∎

COROLLARY 2.2. – If (R, M) is a PVR, then the following conditions are equivalent:

(1) $R' = V = (M: M) = \{x \in T \mid xM \subset M\}$;

(2) Each overring of R is a PVR;

(3) Each overring of R that does not contain an element of the form $1/s$ for some $s \in M$ is a PVR;

(4) For each $u \in V - R$, $R[u]$ is a PVR;

(5) For each $u \in V - R$, $R[u]$ is quasilocal;

(6) Each overring of R is quasilocal.

PROOF. – (1) $\Leftrightarrow$ (2) by [6, Theorem 21]; (2) $\Rightarrow$ (3) trivially; and (3) $\Rightarrow$ (2) by [6, Lemma 20 and Corollary 4]. Moreover, (2) $\Rightarrow$ (6) $\Rightarrow$ (5) trivially; and (5) $\Rightarrow$ (4) by Theorem 2.1. It suffices to prove that (4) $\Rightarrow$ (1). For this, note via the proof of Theorem 2.1 that (4) implies that $(M: M) \subset R'$, while [6, Lemma 17] gives the reverse inclusion. ∎

EXAMPLE 2.3. – (a) Theorem 2.1 does not extend to overrings which are generated by more than one element. In fact, if (R, M) is a PVD and A is a quasilocal overring of R which is contained in $(M: M)$, then A need not be a PVD. For an example, consider $R = \mathbb{Q} + X\mathbb{Q}(s, t)[[X]] = \mathbb{Q} + M$, where s, t, and X are indeterminates and $M = X\mathbb{Q}(s, t)[[X]]$. Observe that $A = \mathbb{Q}[s, t]_{(s, t)} + M$ is a quasilocal overring of R which is contained in $(M: M) = \mathbb{Q}(s, t)[[X]]$, although A is not a PVD.

(b) Not all PVRs satisfy the equivalent conditions in Corollary 2.2. We next illustrate this with an example in which R is an integrally closed PVD. Let t and X be indeterminates and let $V = \mathbb{Q}(t)[[X]] = \mathbb{Q}(t) + M$, where $M = XV$. Then V is a valuation domain, and hence $R = \mathbb{Q} + M$ is a PVD with maximal ideal M and $(M: M) = V$. However, R has an overring, namely $R[t] = \mathbb{Q}[t] + M$, which is not quasilocal.

REMARK 2.4. – The equivalence of (5) and (6) in Corollary 2.2 has the following counterpart for arbitrary rings. Let R be a ring with integral closure R', and let $M \subset R$. Then (R', M) is quasilocal $\Leftrightarrow$ for each $u \in R'$, $(R[u], M)$ is quasilocal $\Leftrightarrow$ each integral overring of R is quasilocal with unique maximal ideal M. For a proof, note first via the incomparability and going-up properties that if (R', M) is quasilocal, then each integral overring of R is quasilocal with unique maximal ideal M. On the other hand, suppose that $(R[u], M)$ is quasilocal for each $u \in R'$. Then if R has distinct maximal ideals M_1 and M_2, pick $v \in M_1 - M_2$ and note, via the going-up property of $R[v] \subset R'$, that $M_1 \cap R[v]$ and $M_2 \cap R[v]$ are distinct maximal ideals of $R[v]$, the desired contradiction.

3. – Strongly prime ideals.

We next study some properties of strongly prime ideals. Recall that a strongly prime ideal of R is comparable under inclusion to each ideal of R [6, Lemma 1(a)], and hence, that $Z(R)$ is a prime ideal if R is a PVR.

LEMMA 3.1. – Let P be a strongly prime ideal of a ring R. Then

(a) P is comparable to $Z(R)$.

(b) $P_{S'}$ is a strongly prime ideal of $R_{S'}$ for any multiplicative subset S' of R disjoint from P.

(c) P_P is a strongly prime ideal of R_P and R_P is a PVR.

(d) If P contains a non-zerodivisor of R, then $P_P = P$.

(e) Each prime ideal $Q \subset P$ of R is strongly prime. Moreover, $(P: P) \subset (Q: Q)$.

PROOF. – (a) This is clear since $Z(R)$ is a union of prime ideals of R [12, page 3] and a strongly prime ideal is comparable to each (prime) ideal of R [6, Lemma 1(a)].

(b) This follows immediately from the definitions.

(c) By part (b) above, P_P is a strongly prime ideal of R_P, and hence R_P is a PVR [6, Theorem 2].

(d) By part (a) above, $Z(R) \subset P$, and hence $R_P \subset R_S$. Let $p/s \in P_P$ with $p \in P$ and $s \in R - P$. Then $P \subset sR$ by [6, Lemma 1(a)]; so $p/s \in P_P \cap R = P$. Hence $P_P = P$.

(e) For the first assertion, just use the proof of [6, Theorem 2]. For the «moreover» statement, we may assume that $Q \neq P$. Let $x \in (P: P)$. Then $xP \subset P$, and hence $xQ \subset P \subset R$. Then $(xQ)P = (xP)Q \subset Q$ yields $xQ \subset Q$ since Q is prime. Thus $x \in (Q: Q)$, and hence $(P: P) \subset (Q: Q)$. ■

We first concentrate on the case when P is a strongly prime ideal which contains a non-zerodivisor of R. In this case, we show that $R \subset R_P \subset (P: P) \subset P^{-1} \subset T$; and that $(P: P) = P^{-1}$ if P is not principal (Theorem 3.6); and in Corollary 3.7(b), we determine when $R_P = (P: P)$.

PROPOSITION 3.2. – Let P be a strongly prime ideal of a ring R which contains a non-zerodivisor of R. Then

(a) $R \subset R_P \subset (P: P) \subset P^{-1} \subset T$.

(b) $P^{-1} \neq T$.

PROOF (a) By Lemma 3.1(a), $Z(R) \subset P$, and hence $R_P \subset T$. Thus we need only show that $R_P \subset (P: P)$. This follows from Lemma 3.1(d) since $R_P \subset (P_P: P_P) = (P: P)$.

(b) Let $s \in P$ be a non-zerodivisor. If $P^{-1} = T$, then $1/s^2 \in P^{-1}$; and hence $1/s = s(1/s^2) \in PP^{-1} \subset R$, a contradiction. Thus $P^{-1} \neq T$. ■

We next give another condition for a prime ideal to be strongly prime. This generalizes [6, Theorem 5].

PROPOSITION 3.3. – Let P be a prime ideal of a ring R. Then

(a) Suppose that $Z(R) \subset P$. Then P is strongly prime if and only if for every $a, b \in R$, either $bR \subset aR$ or $aP \subset bP$.

(b) Let P be a strongly prime ideal. If either P contains a non-zerodivisor or P is a maximal ideal of R, then for every $a, b \in R$, either $bR \subset aR$ or $aP \subset bP$.

PROOF. – (a) Suppose that P is strongly prime. Let $a, b \in R$. If $bR \subset aP$, then $bR \subset aR$. So we may assume that $aP \subset bR$. If $aP \not\subset bP$, then $ap = bc$ for some $p \in P$ and $c \in R - P$. Then c is a non-zerodivisor since $Z(R) \subset P$, and $c \mid p$ since $P \subset cR$ by [6, Lemma 1(a)]. Hence $a \mid b$, and thus $bR \subset aR$.

Conversely, suppose that for every $a, b \in R$, either $bR \subset aR$ or $aP \subset bP$. Let $a, b \in R$. If $aP \subset bP$, then $aP \subset bR$. So we may assume that $bR \subset aR$. Then $b = ac$ for some $c \in R$. If $c \in P$, then $bR \subset aP$. Suppose that $c \notin P$. Then c is a non-zerodivisor since $Z(R) \subset P$. Let $0 \neq p \in P$; then $bp = acp$. We claim that $c \mid p$. If not, then $cP \subset pP$ by hypothesis. Hence $cp = pq$ for some $q \in P$. Thus $p(c - q) = 0$, and hence $c - q \in Z(R) \subset P$. Thus $c \in P$, a contradiction. Hence $c \mid p$ for each $p \in P$, and thus $P \subset cR$. Hence $aP \subset acR = bR$. Thus P is strongly prime.

(b) In either case, $Z(R) \subset P$ by Lemma 3.1(a). Thus part (b) follows from part (a) above. ■

Recall from [6] that an ideal I of a ring R satisfies *property* $(*)$ if whenever $xy \in I$ for some $x, y \in T$, then either $x \in I$ or $y \in I$. It was shown [6, Theorem 14] that if (R, M) is a PVR, then M satisfies property $(*)$. The following proposition is a generalization of that fact.

PROPOSITION 3.4. – Let P be a strongly prime ideal of a ring R. If P contains a non-zerodivisor of R, then P satisfies property $(*)$.

PROOF. – By parts (c) and (d) of Lemma 3.1 and Proposition 3.2(a), R_P is a PVR with maximal ideal $P_P = P$ and total quotient ring T. Thus P satisfies property $(*)$ by [6, Theorem 14]. ■

For our next result, cf. [10, Proposition 1.2] and [6, Lemma 13].

PROPOSITION 3.5. – Let P be a strongly prime ideal of a ring R which contains a non-zerodivisor of R. Then

(a) $T - R \subset U(T)$.

(b) If $x \in T - R$, then $x^{-1}P \subset P$.

Proof. – (a) Let $x = a/b \in T - R$, where $a \in R$ and $b \in R - Z(R)$. Suppose that $a \in Z(R)$. Since P is strongly prime, aR and bP are comparable. If $aR \subset bP$, then $x \in P \subset R$, a contradiction. Thus $bP \subset aR \subset Z(R)$, and hence $b \in Z(R)$ since P contains a non-zerodivisor, again a contradiction. Thus $a \notin Z(R)$; so $x^{-1} = b/a \in T$, and thus $x \in U(T)$.

(b) We have $x(x^{-1}P) = P$; so $x^{-1}P \subset P$ since P satisfies property $(*)$ by Proposition 3.4. ■

Recall that a ring R is a *chained ring* if its ideals are linearly ordered by inclusion (i.e., for every $x, y \in R$, either $x|y$ or $y|x$). Any chained ring is necessarily a PVR [6, Corollary 4]. The following result is motivated by [2, Proposition 4.3] and [4, Proposition 5].

Theorem 3.6. – Let P be a strongly prime ideal of a ring R which contains a non-zerodivisor of R. Then $(P\colon P)$ is a chained ring with maximal ideal P. Moreover, if P is nonprincipal, then $(P\colon P) = P^{-1}$; and if P is principal, then $(P\colon P) = R$.

Proof. – By parts (c) and (d) of Lemma 3.1, R_P is a PVR with maximal ideal $P_P = P$. Since P contains a non-zerodivisor, $(P\colon P) = (P_P\colon P_P)$ is a chained ring with maximal ideal $P_P = P$ by [6, Theorem 8].

For the «moreover» statement, first suppose that P is not principal. Let $x \in P^{-1} - (P\colon P)$. Then $x^{-1} \in T$ by Proposition 3.5(a), and hence $P \subset x^{-1}R$. Since $x^{-1}(xP) = P$ and $x \notin (P\colon P)$, we have $x^{-1} \in P$ since P satisfies property $(*)$ by Proposition 3.4. Thus $P = x^{-1}R$, a contradiction. Hence $(P\colon P) = P^{-1}$. If P is principal, then $P = sR$ for some non-zerodivisor $s \in P$. Thus $(P\colon P) = (sR\colon sR) = R$. ■

Corollary 3.7. – Let P be a strongly prime ideal of a ring R. Then

(a) If P contains a non-zerodivisor prime p of R, then P is maximal, $P = pR$, and R is a chained ring (and thus a PVR).

(b) Suppose that P contains a non-zerodivisor of R. Then $R_P = (P\colon P)$ if and only if R_P is a chained ring.

(c) Let Q be a prime ideal of R properly contained in P. If Q contains a non-zerodivisor of R, then Q is strongly prime and $R_Q = (Q\colon Q)$.

Proof. – (a) Let $y \in R$. Suppose that $y \notin pR$. Then $pP \subset yP$ by Proposition 3.3(b). Hence $p^2 = ym$ for some $m \in R$. Since $p^2 \nmid y$ and p is a non-zerodivisor prime of R, $p^2 | m$. Hence $m = p^2 k$ for some $k \in R$. Thus $p^2 = yp^2 k$, and hence

$yk = 1$. Thus $y \in U(R)$. Hence P is maximal, $P = pR$, and R is a PVR. Thus $R = (P : P)$ is a chained ring by Theorem 3.6.

(b) If $R_P = (P : P)$, then R_P is a chained ring by Theorem 3.6. Conversely, suppose that R_P is a chained ring. Then $R_P \subset (P : P)$ by Proposition 3.2(a). Let $x \in (P : P)$. We may assume that $x \notin P$, and hence x is a unit of $(P : P)$. Thus either x or x^{-1} is in R_P since R_P is a chained ring. If $x^{-1} \in R_P$, then $x^{-1} \in R_P - P$, and hence $x = (x^{-1})^{-1} \in R_P$. Thus $R_P = (P : P)$.

(c) By Lemma 3.1(e) and Theorem 3.6, $(P : P) \subset (Q : Q)$ are chained rings with maximal ideals P and Q, respectively. Thus $R_Q = (R_P)_{Q_P} = (P : P)_{Q_P}$ is a chained ring by [6, Theorem 12]; so $R_Q = (Q : Q)$ by part (b) above. ■

The next result is motivated by [2, Proposition 4.6].

THEOREM 3.8. – The following statements are equivalent for a proper ideal I of a ring R which contains a non-zerodivisor of R:

(1) I is a nonprincipal strongly prime ideal of R.

(2) I^{-1} is a ring and for every $a, b \in R$, the ideals aI and bR are comparable.

PROOF. – (1) ⇒ (2): This is clear by the definition of strongly prime ideal and Theorem 3.6.

(2) ⇒ (1): Let $s \in I$ be a non-zerodivisor of R. The proof of Proposition 3.2(b) shows that I is not principal. We need only show that I is prime. Since $s \in I$ is a non-zerodivisor and for every a, $b \in R$, the ideals aI and bR are comparable, $Z(R) \subset I$. Suppose that $xy \in I$ for some $x, y \in R - I$. Hence $I \subset xR$ and $I \subset yR$ by hypothesis. Since $x, y \in R - I$ and $Z(R) \subset I$, both x and y are non-zerodivisors. Thus $1/x$, $1/y \in I^{-1}$, and hence $1/(xy)^2 \in I^{-1}$ since I^{-1} is a ring. Thus $1/xy = xy/(xy)^2 \in II^{-1} \subset R$, a contradiction. Hence I is prime. ■

We have the following partial converse to Theorem 3.6.

THEOREM 3.9. – Let P be a prime ideal of a ring R such that $B = (P : P)$ is a PVR with maximal ideal $M \in \operatorname{Spec}(R)$. Then

(a) $Z(R) \subset M$.

(b) M, P, and $Z(R)$ are strongly prime ideals of R.

In particular, if $(P : P)$ is a PVR with maximal ideal P, then P is a strongly prime ideal of R and $Z(R) \subset P$.

PROOF. – (a) Let $x \in R - M$. Then $x \in U(B)$. Thus $x \notin Z(R)$, so $Z(R) \subset M$.

(b) Let $a, b \in R$. Since M is a strongly prime ideal of B, the ideals bB

and aM are comparable. If $bB \subset aM$, then $bR \subset aM$. Thus we may assume that $aM \subset bB$. If $aM \not\subset bR$, then $am = bd$ for some $m \in M$ and $d \in B - R$. Thus $d \in U(B)$, and hence $b = a(d^{-1}m) \in aM$. Thus $bR \subset aM$, and hence M is a strongly prime ideal of R. Since $P \subset M$, P is also a strongly prime ideal of R by Lemma 3.1(e). Since $Z(R) \subset M$ and the prime ideals of R contained in M are strongly prime and linearly ordered, $Z(R)$ is a prime ideal of R. Hence $Z(R)$ is a strongly prime ideal of R by Lemma 3.1(e).

The «in particular» statement is immediate. ■

We next consider the case when the strongly prime ideal P does not contain a non-zerodivisor, i.e., when $P \subset Z(R)$. For this case, the next result is analogous to Proposition 3.2.

PROPOSITION 3.10. – Let P be a strongly prime ideal of a ring R such that $P \subset Z(R)$. Then

(a) $P_S = P$.

(b) $(P: P) = T$.

(c) $P = P_S$ is a strongly prime ideal of $R_S = T = (P: P)$.

PROOF. – Let $s \in S$. Then $P \subset sR$ by [6, Lemma 1(a)], and hence $(1/s)\, P \subset R$. Thus $s((1/s)\, P) \subset P$, $s \notin P$, and P a prime ideal yields $(1/s)\, P \subset P$. Hence $P_S = P$ and $(P: P) = T$. That P_S is strongly prime follows from Lemma 3.1(b). ■

THEOREM 3.11. – Let P be a prime ideal of a ring R such that $P \subset Z(R)$. Then

(a) T is a PVR if and only if $Z(R)_S$ is a strongly prime ideal of T.

(b) $(P: P)$ is a PVR with maximal ideal $M \in \operatorname{Spec}(R)$ if and only if $Z(R)$ is a strongly prime ideal of R.

(c) If $(P: P)$ is a PVR with maximal ideal $M \in \operatorname{Spec}(R)$, then $(P: P) = T$, P is a strongly prime ideal of R, and $M = Z(R)$.

PROOF. – Let $Q := Z(R)$.

(a) If T is a PVR, then $T = R_S$ is quasilocal, necessarily with maximal ideal Q_S. Conversely, if Q_S is a strongly prime ideal of T, then T is a PVR by [6, Theorem 2].

(b) If Q is a strongly prime ideal of R, then $T = R_S$ is a PVR with maximal ideal $Q = Q_S$ by Lemma 3.1(c) and Proposition 3.10(a). Thus $P \subset Q$ is also a strongly prime ideal of R by Lemma 3.1(e); so $(P: P) = T$ by Proposition 3.10(b). The converse follows from Theorem 3.9(b).

(c) Suppose that $(P\colon P)$ is a PVR with maximal ideal $M \in \operatorname{Spec}(R)$. Then P is a strongly prime ideal of R by Theorem 3.9(b), and hence $(P\colon P) = T$ by Proposition 3.10(b). By part (a) above, necessarily $M = Q_S = Q_S \cap R = Q$. ■

The next two corollaries summarize our earlier results on when $(P\colon P)$ is a PVR.

COROLLARY 3.12. – Let P be a prime ideal of a ring R. If $(P\colon P)$ is a PVR with maximal ideal $M \in \operatorname{Spec}(R)$, then P and $Z(R)$ are strongly prime ideals of R and either $M = P$ or $M = Z(R)$.

PROOF. – By Theorem 3.9(b), M, P, and $Z(R)$ are strongly prime ideals of R. Thus either $Z(R) \subset P$ or $P \subset Z(R)$ by Lemma 3.1(b). If $P \subset Z(R)$, then $M = Z(R)$ by Theorem 3.11(c). If $Z(R)$ is properly contained in $P(\subset M)$, then $M = P$ by Theorem 3.6. ■

COROLLARY 3.13. – Let P be a prime ideal of a ring R. Then the following statements are equivalent:

(1) $(P\colon P)$ is a PVR with maximal ideal $M \in \operatorname{Spec}(R)$;

(2) P and $Z(R)$ are strongly prime ideals of R.

PROOF. – $(1) \Rightarrow (2)$ by Theorem 3.9(b).
$(2) \Rightarrow (1)$: By Lemma 3.1(a), P and $Z(R)$ are comparable. If $P \subset Z(R)$, then we are done by Theorem 3.11(b). If $Z(R)$ is properly contained in P, then we are done by Theorem 3.6. ■

QUESTION 3.14. – Let P be a strongly prime ideal of a ring R such that $P \subset Z(R)$ and $(P\colon P)$ $(= T)$ is a PVR. Then T has maximal ideal $Z(R)_S$ and $Z(R)$ is a prime ideal of R. Is $Z(R)$ also a strongly prime ideal of R?

Any PVD which is not a field gives an example of a PVR (R, M) for which $\operatorname{nil}(R) = Z(R) \neq M$. In [6, Example 10(b)], we constructed a PVR (R, M) with $\operatorname{nil}(R) \neq Z(R) = M$. These examples raise the question whether there exists a PVR (R, M) for which $\operatorname{nil}(R)$ is neither $Z(R)$ nor M. In Example 3.16(c), we show that such behavior is possible. In the next proposition, we give a necessary and sufficient condition for certain rings R to have $\operatorname{nil}(R) = Z(R)$.

PROPOSITION 3.15. – Let R be a ring such that either R is quasilocal or $\operatorname{nil}(R)$ is a (minimal) prime ideal of R. Then $Z(R) = \operatorname{nil}(R)$ if and only if for every $x \in Z(R)$ there exists an integer $k \geqslant 1$ such that $x^k R = x^{k+1} R$.

PROOF. – We need only prove the «if» assertion. Suppose that there is an

$x \in Z(R) - \mathrm{nil}(R)$. Then $x^k = x^{k+1}m$ for some $m \in R$ and some integer $k \geq 1$. Hence $x^k(1 - xm) = 0 \in \mathrm{nil}(R)$. If $\mathrm{nil}(R)$ is prime, then $1 - xm \in \mathrm{nil}(R)$ since $x \notin \mathrm{nil}(R)$. Hence $xm = 1 - (1 - xm) \in U(R)$, and thus $x \in U(R)$, a contradiction. If R is quasilocal with maximal ideal M, then $x \in M$, and hence $1 - xm \in U(R)$. Thus $x^k = 0$, so $x \in \mathrm{nil}(R)$, again a contradiction. Thus $Z(R) = \mathrm{nil}(R)$. ■

We end the paper with several examples. In particular, Example 3.16(c) shows that if R is a PVR with maximal ideal M, then $Z(R)$ can be $\mathrm{nil}(R)$, M, or a prime ideal properly contained between these two ideals.

EXAMPLE 3.16. – (a) ([6, Example 10(a)]) Let k be a field and X and Y indeterminates. Then $R = k[X, Y]/(X^2, XY, Y^2) = k[x, y]$ is a zero-dimensional PVR with (strongly prime) maximal ideal $M = Z(R) = (x, y)$, and $(M : M) = R$ is not a chained ring. Thus the non-zerodivisor hypothesis is needed in Theorem 3.6.

(b) Let W be a valuation domain with maximal ideal N. For any $0 \neq x \in N$, $R = W/xW$ is a PVR [6, Corollary 3] with maximal ideal $M = Z(R) = N/xW$ and $\mathrm{nil}(R) = Q/xW$, where $Q = \sqrt{xW}$ is the (unique) prime ideal of W minimal over xW. To see that $Z(R) = N/xW$, observe that for any $m \in N - xW$, then $x = rm$ for some $r \notin xW$, and hence $(r + xW)(m + xW) = 0$ in W/xW with $r + xW$ nonzero. This example generalizes [6, Example 10(b)].

(c) Let W be a valuation domain with maximal ideal N and let $0 \neq x \in N$. Then by part (b) above, $W^* = W/xW$ is a chained ring with maximal ideal $N^* = N/xW = Z(W^*)$, $\mathrm{nil}(W^*) = \sqrt{xW}/xW$, and residue field $k = W^*/N^* = W/N$. Let $\pi : W^* \to k$ be the natural surjection, let D be a valuation domain with maximal ideal P and quotient field k, and let $R = \pi^{-1}(D)$. Then R is a chained ring with maximal ideal $M = \pi^{-1}(P) \supset N^*$, $\mathrm{nil}(R) = \mathrm{nil}(W^*)$, and $Z(R) = Z(W^*) = N^*$. (Also note that $R = \mu^{-1}(D)/xW$, where $\mu : W \to k$ is the natural surjection; so $\mu^{-1}(D) \subset W$ is a valuation domain.)

By standard gluing techniques (cf. [9, Corollary 1.5]), $\mathrm{Spec}(R)$ is order-isomorphic to the result of gluing $\mathrm{Spec}(D)$ «above» $\mathrm{Spec}(W^*)$, where 0 in $\mathrm{Spec}(D)$ is identified with N^* in $\mathrm{Spec}(W^*)$. Thus for any integers i and n with $1 \leq i \leq n$, there is an $(n-1)$-dimensional chained ring (R, M) with distinct prime ideals $\mathrm{nil}(R) = M_1 \subset M_2 \subset \ldots \subset M_n = M$ such that $Z(R) = M_i$.

More generally, let $(I, \leq)$ be any set which can be realized as the spectrum of some valuation domain (i.e., by [13, Corollary 3.6], I is linearly ordered and satisfies properties (K1) and (K2) (cf. [12, pages 6-7])). Let m be the minimum element of I, L the maximum element of I, and $i \in I$ with $m \leq i \leq L$. By the above construction, there is a chained ring (and hence a PVR) (R, M) with $\mathrm{Spec}(R)$ order-isomorphic to I, where $\mathrm{nil}(R) \leftrightarrow m$, $Z(R) \leftrightarrow i$, and $M \leftrightarrow L$.

* * *

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